2 — Matrices and Determinants

2.1 Definition of Matrix

Definition 2.1.1 A matrix is defined as a rectangular array of numbers, enclosed by brackets.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

R The numbers in the array are called the elements of the matrix.

Some examples of matrices are:

$$\begin{bmatrix} -1 & 3 & 8 \\ -3 & 4 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 7 & 11 \\ 0 & 1 \end{bmatrix}, \begin{pmatrix} -1 & 3 & 8 \\ 2 & 7 & 11 \\ -3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} -1.3 & 0.8 \end{pmatrix}, A = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

Note:

- If a matrix A has m rows(horizontals) and n columns(verticals), then the size (or dimension) of the matrix A is $m \times n$ (*read as 'm by n'*).
- a_{ij} is the element that appears in the i^{th} row and in the j^{th} column.
- $A = (a_{ij})_{m \times n}$ is an $m \times n$ matrix.

Example 2.1 Consider $A = \begin{pmatrix} 0 & 3 & 8 \\ 2 & 7 & 11 \end{pmatrix}$.
The size of matrix A is 2×3 because it has two rows and three columns.
The elements
$a_{11} = 0$, First row, first column, $\quad a_{23} = 11$ Second row, third column ■

Definition 2.1.2 Two matrices are equal if they have the same size and their corresponding elements are equal.

■ **Example 2.2** Let $A = \begin{bmatrix} 3 & -5 & x \\ 2 & y+1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 & 2x-1 \\ 2 & 5 & 3 \end{bmatrix}$. Find the values of x and y such that $A = B$ ■

Solution: For the two matrices to be equal, we must have corresponding entries equal, so

$$x = 2x - 1 \implies x = 1 \quad a_{13} = b_{13}$$
$$y + 1 = 5 \implies y = 4 \quad a_{22} = b_{22}$$

■ **Example 2.3** Consider the two matrices given below

$$A = \begin{pmatrix} -1 & 3 & 8 \\ 2 & 7 & 11 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 4 & 8 \\ 2 & 7 & 11 \end{pmatrix}.$$

■

Solution: The size of the two matrix is 2×3. Since $a_{12} = 3 \neq 4 = b_{12}$, we can say that $A \neq B$

2.1.1 Types of Matrices

1. **Row Matrix:** A matrix which has exactly one row $(1 \times n)$ called a row matrix (row vector)
   Example: $A = \begin{pmatrix} 2 & 5 & 6 \end{pmatrix}$ is a 1×3 row matrix.
2. **Column Matrix:** A matrix which has exactly one column $(m \times 1)$ matrix is called a column matrix (or column vector).
   Example: $A = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is a 3×1 column matrix.
3. **Square Matrix:**A matrix in which the number of rows is equal to the number of columns is called a square matrix.$(n \times n)$
   $$B = \begin{pmatrix} -1 & 4 & 8 \\ 2 & 7 & 11 \\ 3 & -6 & 0 \end{pmatrix}.$$
   is a 3×3 square matrix.
4. **Null or zero matrix:** A matrix each of whose elements is zero is called a zero matrix.
   Example: $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a 2×3 zero matrix.
5. **Diagonal matrix:**is a square matrix whose every element other than diagonal elements is zero. The matrix
   $$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{pmatrix},$$
   are example of diagonal matrix.

   R The elements a_{ii} are called diagonal elements of a square matrix (a_{ij}).

6. **Scalar matrix:** A scalar matrix is a diagonal matrix whose diagonal elements are equal
   $$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

7. **Identity matrix:** is a diagonal matrix whose diagonal elements are equal to 1 (units)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

8. **Triangular matrix:**
A square matrix whose elements $a_{ij} = 0$ when ever $i < j$ is called a lower triangular matrix. And square matrix whose elements $a_{ij} = 0$ when ever $i > j$ is called an upper triangular matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 11 & 0 & 0 \\ 3 & 2 & 4 & 0 \\ 0 & -3 & 6 & 12 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 1.5 & 1 \end{pmatrix}$$

are lower triangular matrix. And

$$\begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 2 & 8 & 5 \\ 0 & 0 & 7 & -9 \\ 0 & 0 & 0 & 12 \end{pmatrix}, \begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 5 \\ 0 & 10 & 8 \\ 0 & 0 & -1 \end{pmatrix}$$

are upper triangular matrix

2.2 Matrix Addition and Scalar Multiplication

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices with the same size, say $m \times n$. The sum of A and B, written $A + B$, is the matrix obtained by adding corresponding elements from A and B. That is, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

then,

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{m1} + b_{mn} & a_{m2} + b_{m2} & \dots & a_{m1} + b_{mn} \end{pmatrix}$$

The product of the matrix A by a scalar k, written kA, is the matrix obtained by multiplying each element of A by k. That is,

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}$$

The matrix $-A$ is called the negative of the matrix A, and the matrix $A - B$ is called the difference of A and B.

Subtraction is performed on matrices of the same size by subtracting corresponding elements. Thus, $A - B = A + (-B)$

R The sum of matrices with different sizes is not defined.

Example 2.4 Let $A = \begin{bmatrix} -1 & 3 \\ 7 & 11 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 4 & 1 \\ 10 & -6 \end{bmatrix}$. Find

(a) $A+B$ (b) $2A$ (c) $3A-2B$

Solution:

(a) $A+B = \begin{bmatrix} -1 & 3 \\ 7 & 11 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 4 & 1 \\ 10 & -6 \end{bmatrix} = \begin{bmatrix} -1+3 & 3+5 \\ 7+4 & 11+1 \\ 0+10 & 1+(-6) \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 11 & 12 \\ 10 & -5 \end{bmatrix}$

(b) $2A = 2\begin{bmatrix} -1 & 3 \\ 7 & 11 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 14 & 22 \\ 0 & 2 \end{bmatrix}$

(c) $3A-2B = 3\begin{bmatrix} -1 & 3 \\ 7 & 11 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 5 \\ 4 & 1 \\ 10 & -6 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 21 & 33 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 10 \\ 8 & 2 \\ 20 & -12 \end{bmatrix} = \begin{bmatrix} -9 & -1 \\ 13 & 31 \\ -20 & 15 \end{bmatrix}$

Properties of Matrix Addition and Scalar Multiplication

If $A, B,$ and C are any $m \times n$ matrices and if O is the zero $m \times n$ matrix, and $k,\ l$ are any real number, then the following hold:

(a) Associative law: $A+(B+C) = (A+B)+C$
(b) Commutative law: $A+B = B+A$
(c) Additive identity law: $A+O = O+A = A$
(d) Additive inverse law: $A+(-A) = -A+A = O$
(e) Distributive law: $k(A+B) = kA+kB$
Distributive law: $(l+k)A = lA+kA$
(f) Scalar unit $1A = A$
(g) Scalar zero $0A = O$

2.2.1 Transpose Matrix

Definition 2.2.1 The transpose of a matrix A, written A^t, is the matrix obtained by interchanging the columns and rows of A. That is, if $A = (a_{ij})$ is $m \times n$ matrix, then $A^t = (b_{ij})$ is $n \times m$ matrix where $(b_{ij}) = (a_{ji})$

Example 2.5 Let $A = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 5 & -9 \end{pmatrix}$. Then $A^t = \begin{pmatrix} 4 & 2 \\ 2 & 5 \\ 3 & -9 \end{pmatrix}$

Properties

(a) $(A+B)^t = A^t + B^t$
(b) $(A^t)^t = A$
(c) $(kA)^t = kA^t$, k scalar
(d) $(AB)^t = B^t A^t$

R A square matrix A is symmetric if $A^t = A$.
A square matrix A is skew symmetric if $A^t = -A$.

■ **Example 2.6** Let $A = \begin{pmatrix} -2 & 5 & 4 \\ 5 & 9 & 12 \\ 4 & 12 & -7 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -5 & 9 \\ 5 & 0 & 6 \\ -9 & -6 & 0 \end{pmatrix}$. Find A^t, B^t ■

Solution: $A^t = \begin{pmatrix} -2 & 5 & 4 \\ 5 & 9 & 12 \\ 4 & 12 & -7 \end{pmatrix} = A \implies$, Matrix A is symmetric

$B^t = \begin{pmatrix} 0 & 5 & 9 \\ -5 & 0 & -6 \\ 9 & 6 & 0 \end{pmatrix} = -B \implies$, Matrix B is skew symmetric

Theorem 2.2.1 Let A be a square matrix. Then
(a) $A + A^t$ is symmetric.
(b) $A - A^t$ is skew symmetric.

■ **Example 2.7** Let $A = \begin{pmatrix} 4 & 3 & 6 \\ -1 & 0 & 7 \\ 3 & -2 & 1 \end{pmatrix}$ ■

Solution: $A^t = \begin{pmatrix} 4 & -1 & 3 \\ 3 & 0 & -2 \\ 6 & 7 & 1 \end{pmatrix}$, then

$B = A + A^t = \begin{pmatrix} 4 & 3 & 6 \\ -1 & 0 & 7 \\ 3 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -1 & 3 \\ 3 & 0 & -2 \\ 6 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 9 \\ 2 & 0 & 5 \\ 9 & 5 & 2 \end{pmatrix} = B^t$, which is symmetric

$C = A - A^t = \begin{pmatrix} 4 & 3 & 6 \\ -1 & 0 & 7 \\ 3 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & -1 & 3 \\ 3 & 0 & -2 \\ 6 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 3 \\ -4 & 0 & 9 \\ -3 & -9 & 0 \end{pmatrix} = -C^t$, which is skew symmetric

2.3 Matrix Multiplication

Definition 2.3.1 Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. The product matrix AB is the $m \times k$ matrix whose entry in the i th row and j th column is the product of the i th row of A and the j th column of B.

R The number of columns of A must equal the number of rows of B in order to get the product matrix AB.

Example 2.8 Let $A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 6 & -2 \\ -4 & 2 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 10 & 2 \\ 4 & 3 \end{pmatrix}$. Find

(a) AB (b) BA (c) BC (d) CA (e) AC

Solution:

(a) Here, matrix A is 2×3 and matrix B is 3×2, so matrix AB can be found and will be 2×2.

$$AB = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -4 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.6 + 3.-4 + 4.0 & 2.-2 + 3.2 + 4.1 \\ -1.6 + 2.-4 + 2.0 & -1.-2 + 2.2 + 2.1 \end{pmatrix}$$

$$= \begin{pmatrix} 12 + -12 + 0 & -4 + 6 + 4 \\ -6 + -8 + 2 & 2 + 4 + 2 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ -12 & 8 \end{pmatrix}$$

(b) Here, matrix B is 3×2 and matrix A is 2×3, so matrix BA can be found and will be 3×3.

$$BA = \begin{pmatrix} 6 & -2 \\ -4 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 6 \times 2 + -2 \times -1 & 6 \times 3 + -2 \times 2 & 6 \times 4 + -2 \times 2 \\ -4 \times 2 + 2 \times -1 & -4 \times 3 + 2 \times 2 & -4 \times 4 + 2 \times 2 \\ 0 \times 2 + 1 \times -1 & 0 \times 3 + 1 \times 2 & 0 \times 4 + 1 \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 14 & 20 \\ -10 & -8 & -12 \\ -1 & 2 & 2 \end{pmatrix}$$

(c) Here, matrix B is 3×2 and matrix C is 2×2, so matrix BC can be found and will be 3×2.

$$BC = \begin{pmatrix} 6 & -2 \\ -4 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 10 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 6 \times 10 + -2 \times 4 & 6 \times 2 + -2 \times 3 \\ -4 \times 10 + 2 \times 4 & -4 \times 2 + 2 \times 3 \\ 0 \times 10 + 1 \times 4 & 0 \times 2 + 1 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 52 & 6 \\ -32 & -2 \\ 4 & 3 \end{pmatrix}$$

(d) Here, matrix C is 2×2 and matrix A is 2×3, so matrix CA can be found and will be 2×3.

$$CA = \begin{pmatrix} 10 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 10 \times 2 + 2 \times -1 & 10 \times 3 + 2 \times 2 & 10 \times 4 + 2 \times 2 \\ 4 \times 2 + 3 \times -1 & 4 \times 3 + 3 \times 2 & 4 \times 4 + 3 \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 34 & 44 \\ 5 & 18 & 22 \end{pmatrix}$$

(e) Since the number of column in matrix A (2×3) is not equal to the number of rows in matrix C (2×2), the product AC is not defined.

Properties of Matrix Multiplication

Let A be a matrix of dimension $m \times k$, let B be a matrix of dimension $k \times r$, and let C be a matrix of dimension $r \times n$

(a) Matrix multiplication is not commutative. That is, in general, $AB \neq BA$
(b) Matrix multiplication is associative. That is, $A(BC) = (AB)C$
(c) Distributive property: $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$
(d) $k(AB) = (kA)B = A(kB)$, where k is a scalar.
(e) If A is a square matrix of dimension $n \times n$, then $AI_n = I_nA = A$, where I_n is $n \times n$ identity matrix
(f) If A is $m \times n$, then $AI_n = A = I_mA$

R If A and B are matrices such that the products AB and BA exist, then AB may not equal BA. The cancellation law do not hold for matrix multiplication. That is, if $AB = AC$ then it is not true in general that $B = C$.
If a product AB is the zero matrix, we cannot conclude in general that either $A = 0$ or $B = 0$

Example 2.9 Let $A = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -3 & -2 \\ 0 & 2 & 0 \\ 7 & 8 & 1 \end{bmatrix}$. Show that $A(BC) = (AB)C, AI_2 = I_2A = A$ ■

Solution: Since matrix A is 2×2, matrix B is 2×3 and matrix C is 3×3, the product $A(BC) = (AB)C$ can be found and will be 2×3

$$\begin{aligned}
AB &= \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 17 & -2 \\ 19 & 5 & 30 \end{bmatrix} \\
BC &= \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & 6 \end{bmatrix} \begin{bmatrix} 4 & -3 & -2 \\ 0 & 2 & 0 \\ 7 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 19 & 0 \\ 58 & 34 & -2 \end{bmatrix} \\
A(BC) &= \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 18 & 19 & 0 \\ 58 & 34 & -2 \end{bmatrix} = \begin{bmatrix} -26 & 27 & 4 \\ 286 & 193 & -8 \end{bmatrix} = \begin{bmatrix} -3 & 17 & -2 \\ 19 & 5 & 30 \end{bmatrix} \begin{bmatrix} 4 & -3 & -2 \\ 0 & 2 & 0 \\ 7 & 8 & 1 \end{bmatrix} = (AB)C
\end{aligned}$$

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AI_2$$

Example 2.10 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$. Find AB, BA ■

Solution: $AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 2 \times -1 & 1 \times -4 + 2 \times 2 \\ 3 \times 2 + 6 \times -1 & 3 \times -4 + 6 \times 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$BA = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + -4 \times 3 & 2 \times 2 + -4 \times 6 \\ -1 \times 1 + 2 \times 3 & -1 \times 2 + 2 \times 6 \end{bmatrix} = \begin{bmatrix} -10 & -20 \\ 5 & 10 \end{bmatrix}$$

2.4 Elementary Row Operations and Echelon Form

Let A be an $m \times n$ matrix. The elementary row operation on A is

(a) Interchanging two rows of a matrix . If we interchange the i^{th} row with the j^{th} row, then we usually denote the operation as $R_i \longleftrightarrow R_j$

(b) Multiply the elements of a row by a non-zero constant. If the i^{th} row is multiplied by α then we usually denote this operation as $R_i \longrightarrow \alpha R_i$

(c) Add a multiple of the elements of one row to the corresponding elements of another row i.e., $R_i \longrightarrow R_i + \alpha R_j$

Definition 2.4.1 Two matrices are equivalent written as $A \sim B$ if one can be obtained from the other by a sequence of elementary row operations.

■ **Example 2.11** Let $A = \begin{pmatrix} 2 & -3 & 6 \\ 1 & 1 & 4 \\ 5 & -4 & 1 \end{pmatrix}$ ■

Solution:

$$A \sim \begin{pmatrix} 2 & -3 & 6 \\ 4 & 4 & 16 \\ 5 & -4 & 1 \end{pmatrix} (R_2 \longrightarrow 4R_2) \sim \begin{pmatrix} 2 & -3 & 6 \\ 5 & -4 & 1 \\ 4 & 4 & 16 \end{pmatrix} (R_2 \longrightarrow R_3) \sim \begin{pmatrix} 2 & -3 & 6 \\ 5 & -4 & 1 \\ 0 & 10 & 4 \end{pmatrix} (R_3 \longrightarrow R_3 - 2R_1)$$

Definition 2.4.2 A matrix is in a row echelon form if it satisfays the following conditions

(a) The first non-zero entry in each row is a 1 (called a Leading 1).

(b) If a column contains a leading 1, then every entry of the column below the leading 1 is a zero.

(c) As we move downwards through the rows of the matrix, the leading 1's move from left to right.

(d) Any row (if any) consisting of entirely of zeros appears at the bottom of the matrix.

■ **Example 2.12** The following matrices are in row echelon form

$$A = \begin{pmatrix} 1 & \frac{-3}{5} & \frac{4}{5} \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 14 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -13 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

■

■ **Example 2.13** The following matrices are not in row echelon form

$$E = \begin{pmatrix} 1 & 2 & 14 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & 19 \end{pmatrix}, \quad G = \begin{pmatrix} 4 & 5 & 6 \\ 0 & 6 & 4 \\ 0 & 3 & 3 \end{pmatrix}$$

■

Example 2.14 Let A be the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$$

Reduce A to row-echelon form.

Solution:

1. Get a 1 as the first entry of Row 1. Done.
2. Use this first leading 1 to clear out column 1 as follows :
$$A \sim \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 3 & 1 & -2 & 8 \\ 0 & -2 & 3 & 5 & 2 \end{pmatrix}$$
3. Get a leading 1 as the second entry of Row 2, for example as follows:
$$A \sim \xrightarrow{R_2 \to R_2 + R_3} \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & -2 & 3 & 5 & 2 \end{pmatrix}$$
4. Use this leading 1 to clear out whatever appears below it in Column 2
$$A \sim \xrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 11 & 11 & 22 \end{pmatrix}$$
5. Get a leading 1 in Row 3:
$$\xrightarrow{R_3 \to \frac{1}{11} R_3} \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$
This matrix is now in row-echelon form.

Example 2.15 Let H be the matrix

$$H = \begin{pmatrix} 1 & -1 & 1 & 4 \\ 4 & 2 & -2 & 4 \\ 1 & -3 & 5 & 6 \end{pmatrix}$$

Reduce H to row-echelon form.

Solution:

1. Get a 1 as the first entry of Row 1. Done.
2. Use this first leading 1 to clear out column 1 as follows :
$$H \sim \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 4R_1} \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 6 & -6 & -12 \\ 0 & -2 & 4 & 4 \end{pmatrix}$$
3. Get a leading 1 as the second entry of Row 2, for example as follows:
$$H \sim \xrightarrow{R_2 \to \frac{1}{6} R_2} \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 4 & 4 \end{pmatrix}$$
4. Use this leading 1 to clear out whatever appears below it in Column 2
$$H \sim \xrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$
5. Get a leading 1 in Row 3:
$$\xrightarrow{R_3 \to \frac{1}{2} R_3} \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
This matrix is now in row-echelon form.

Definition 2.4.3 A matrix in a row echelon form is said to be in reduced row echelon form if all entries in any column containing the leading 1 is zero.

■ **Example 2.16** The following matrices are in reduced row echelon form

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ B=\begin{pmatrix}0&1&4&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\ C=\begin{pmatrix}1&-13&0&0\\0&0&1&5\\0&0&0&0\end{pmatrix},\ D=\begin{pmatrix}1&0&2\\0&1&0\\0&0&0\end{pmatrix}$$ ■

■ **Example 2.17** Let A be the matrix

$$A=\begin{pmatrix}1&-1&1&4\\4&2&-2&4\\1&-3&5&6\end{pmatrix}$$

Reduce A to reduced row-echelon form. ■

Solution:

1. Get a 1 as the first entry of Row 1. Done.
2. Use this first leading 1 to clear out column 1 as follows :
$$A\sim\xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-4R_1}\begin{pmatrix}1&-1&1&4\\0&6&-6&-12\\0&-2&4&4\end{pmatrix}$$
3. Get a leading 1 as the second entry of Row 2, for example as follows:
$$A\sim\xrightarrow{R_2\to\frac{1}{6}R_2}\begin{pmatrix}1&-1&1&4\\0&1&-1&-2\\0&-2&4&4\end{pmatrix}$$
4. Use this leading 1 to clear out whatever appears below it in Column 2
$$A\sim\xrightarrow{R_3\to R_3+2R_2}\begin{pmatrix}1&-1&1&4\\0&1&-1&-2\\0&0&2&0\end{pmatrix}$$
5. Get a leading 1 in Row 3:
$$\xrightarrow{R_3\to\frac{1}{2}R_3}\begin{pmatrix}1&-1&1&4\\0&1&-1&-2\\0&0&1&0\end{pmatrix}$$
6. Make zero above the leading 1 in column 2
$$\xrightarrow{R_1\to R_1+R_2}\begin{pmatrix}1&0&0&2\\0&1&-1&-2\\0&0&1&0\end{pmatrix}$$
7. Make zero above the leading 1 in column 3
$$\xrightarrow{R_2\to R_2+R_3}\begin{pmatrix}1&0&0&2\\0&1&0&-2\\0&0&1&0\end{pmatrix}$$
This matrix is now in reduced row-echelon form.

2.4.1 Rank of a matrix

Definition 2.4.4 Let A be an $m \times n$ matrix. Let A_R be the row echelon form of A. The rank $\rho(A)$ is the number of non-zero rows of the row echelon form of A (A_R).

$$\rho(A) \le \min(m,n)$$

■ **Example 2.18** Find the rank of the following matrices.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 9 & 1 \\ 1 & 5 & 1 \end{pmatrix}$$ ■

Solution: (a) $\text{rank}(A) = 3$

(b) Write the augmented matrix and apply elementary row operations

$$B|I = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 9 & 1 \\ 1 & 5 & 1 \end{pmatrix} \sim \xrightarrow[R_2 \longrightarrow R_2 - 2R_1]{R_3 \longleftrightarrow R_2 - 3R_1} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, $\text{Rank}(B) = 2$

2.5 Determinant of a Matrix and its Properties

The determinant of a square matrix is a single number that results from performing a specific operation on the array. The determinant of a matrix A is denoted as $\det(A)$ or $|A|$.

Determinant of order two

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a square matrix of order two. Then

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of order three

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a 3×3 matrix. Then

$$\begin{aligned} \det(A) &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \end{aligned}$$

■ **Example 2.19** Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 2 \\ 0 & -4 & 1 \end{pmatrix}$$ ■

Solution: $\det(A) = 1(3(1) - (-4)2) - 2(-1(1) - 0(2)) + 4((-1)(-4) - 0(3)) = 27$

Properties of determinants

1. If matrix B results from matrix A by interchanging two rows (columns) of A, then
$$\det(B) = -\det(A)$$
2. If two rows (columns) of A are equal, then $\det(A) = 0$
3. The determinant of the transpose of A is equal to the determinant of the given matrix A
$$\det(A^t) = \det(A)$$
4. If a row (column) of A consists entirely of zeros, then $\det(A) = 0$.

5. If B is obtained from A by multiplying a row (column) of A by a real number c, then
$$\det(B) = c\det(A)$$
6. If to any row (or column) is added k times the corresponding elements of another row (or column), the determinant remains unchanged.
7. The determinant of a product of two matrices is the product of their determinants; that is $\det(AB) = \det(A)\det(B)$.
8. If c is a real number and A is $n \times n$ matrix, then $\det(cA) = c^n \det(A)$.
9. The determinant of a diagonal matrix is the product of the diagonal elements.
10. The determinant of identify matrix is 1.

2.6 Inverse of a Matrix

Definition 2.6.1 Let A be a matrix of dimension $n \times n$. A matrix B of dimension $n \times n$ is called the inverse of A if

$$AB = BA = I_n$$

where, I_n is $n \times n$ identity matrix. We denote the inverse of a matrix A, if it exists, by A^{-1}.

■ **Example 2.20** Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$.

Show that matrix A and B are inverse of each other. ■

Solution: Since $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = BA$, then $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$

R

- Only square matrices possibly have an inverse.
- A non-square matrix has no inverse.
- The inverse of a square matrix, if it exists, is unique.
- A square matrix that does not have an inverse is called singular.
- If A is invertible, then $\det(A) \neq 0$ and $\det(A^{-1}) = \dfrac{1}{\det(A)}$

An example of a singular matrix is given by
$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
If B had an inverse given by
$$B^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
where a, b, c, and d are some appropriate numbers, then by the definition of an inverse we would have $BB^{-1} = I$; that is,
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
which implies that $0 = 1$—an impossibility! This contradiction shows that B does not have an inverse.

Properties of Invertible matrices

1. $(A^t)^{-1} = (A^{-1})^t$
2. $(A^{-1})^{-1} = A$
3. $(AB)^{-1} = B^{-1}A^{-1}$

2.6.1 Gauss–Jordan elimination method

To find the inverse of a matrix A by using Gauss-Jordan elimination method the operations are:
Step 1 Augment the matrix A with identity matrix. i.e., Write in the form $A|I$.
Step 2 Reduce the augmented matrix in to reduced row-echelon form.

■ **Example 2.21** Find the inverse of $A = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$ ■

Solution: Write the augmented matrix
$A|I = \left(\begin{array}{cc|cc} 7 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right)$
Apply elementary row operations

$$A|I \sim \xrightarrow{R_2 \longrightarrow 7R_2 - 2R_1} \left(\begin{array}{cc|cc} 7 & 3 & 1 & 0 \\ 0 & 1 & -2 & 7 \end{array}\right) \sim \xrightarrow{R_1 \longrightarrow R_1 - 3R_2} \left(\begin{array}{cc|cc} 7 & 0 & 7 & -21 \\ 0 & 1 & -2 & 7 \end{array}\right)$$
$$\sim \xrightarrow{R_1 \longrightarrow \frac{1}{7}R_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 7 \end{array}\right)$$
$$\text{Thus,} \quad A^{-1} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$$

■ **Example 2.22** Find the inverse of $A = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 1 & -1 \\ -5 & -2 & 4 \end{pmatrix}$ ■

Solution: Write the augmented matrix
$A|I = \left(\begin{array}{ccc|ccc} 3 & 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -5 & -2 & 4 & 0 & 0 & 1 \end{array}\right)$
Apply elementary row operations

$$A|I \sim \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 3 & 1 & -2 & 1 & 0 & 0 \\ -5 & -2 & 4 & 0 & 0 & 1 \end{array}\right) \sim \xrightarrow[R_3 \dashrightarrow R_3 + 5R_1]{R_2 \dashrightarrow R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & -3 & 0 \\ 0 & 3 & -1 & 0 & 5 & 1 \end{array}\right)$$
$$\sim \xrightarrow{R_2 \dashrightarrow R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 3 & -1 & 0 & 5 & 1 \end{array}\right) \sim \xrightarrow{R_3 \dashrightarrow -R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 & 2 \end{array}\right)$$
$$\sim \xrightarrow{R_1 \dashrightarrow R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 & 2 \end{array}\right) \sim \xrightarrow{R_1 \dashrightarrow R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 & 2 \end{array}\right)$$

Hence, $A^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

■ **Example 2.23** Find the inverse of $B = \begin{pmatrix} 2 & 4 & -2 \\ -4 & -6 & 1 \\ 3 & 5 & -1 \end{pmatrix}$ ■

Solution: Write the augmented matrix and apply elementary row operations

$$B|I = \left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ -4 & -6 & 1 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{array}\right) \sim \xrightarrow[R_3 \longrightarrow 2R_3 - 3R_1]{R_2 \longleftrightarrow R_2 + 2R_1} \left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 2 & -3 & 2 & 1 & 0 \\ 0 & -2 & 4 & -3 & 0 & 2 \end{array}\right)$$

$$\sim \xrightarrow{R_3 \longrightarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 2 & -3 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \sim \xrightarrow[R_2 \longrightarrow R_2 + 3R_3]{R_1 \longrightarrow R_1 + 2R_3} \left(\begin{array}{ccc|ccc} 2 & 4 & 0 & -1 & 2 & 4 \\ 0 & 2 & 0 & -1 & 4 & 6 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$\sim \xrightarrow{R_1 \longrightarrow R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -6 & -8 \\ 0 & 2 & 0 & -1 & 4 & 6 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \sim \xrightarrow[R_2 \longrightarrow \frac{1}{2}R_2]{R_1 \longrightarrow \frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -3 & -4 \\ 0 & 1 & 0 & -\frac{1}{2} & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

Thus, $B^{-1} = \begin{pmatrix} \frac{1}{2} & -3 & -4 \\ -\frac{1}{2} & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}$

R If there is a row to the left of the vertical line in the augmented matrix containing all zeros, then the matrix does not have an inverse.

■ **Example 2.24** Find the inverse of $A = \begin{bmatrix} 1 & 3 & -4 \\ -2 & -6 & 8 \\ 5 & -2 & 1 \end{bmatrix}$ if exist. ■

Solution: Write the augmented matrix

$$A|I = \left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ -2 & -6 & 8 & 0 & 1 & 0 \\ 5 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

Apply elementary row operations

$$A|I \sim \xrightarrow[R_3 \longrightarrow R_3 - 5R_1]{R_2 \longrightarrow R_2 + 2R_1} \left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & -17 & 21 & -5 & 0 & 1 \end{array}\right) \sim \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & -17 & 21 & -5 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{array}\right)$$

Since there is a row to the left of the vertical line in the augmented matrix containing all zeros, then the matrix A does not have an inverse.

2.6.2 Cofactor matrix and adjoint

Definition 2.6.2 Let $A = (a_{ij})$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1)\times(n-1)$ sub matrix of A obtained by deleting the i^{th} row and j^{th} column of A. The determinant $\det(M_{ij})$ (is called the minor of a_{ij}. The cofactor A_{ij} of a_{ij} is defined as

$$A_{ij} = (-1)^{i+j}\det(M_{ij}).$$

The cofactor of an $n \times n$ matrix A is the matrix

$$B = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}$$

Definition 2.6.3 The adjoint of a matrix A , written adj$A$, is the transpose of the cofactor matrix B.

Example 2.25 Find the adjoint of

$$A = \begin{pmatrix} 3 & -3 & -2 \\ 1 & -1 & -1 \\ -3 & 4 & 2 \end{pmatrix}$$

Solution:

$A_{11} = |M_{11}| = \begin{vmatrix} -1 & -1 \\ 4 & 2 \end{vmatrix} = 2$, $A_{12} = -|M_{12}| = -\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = 1$, $A_{13} = |M_{13}| = \begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} = 2$

$A_{21} = -|M_{21}| = -\begin{vmatrix} -3 & -2 \\ 4 & 2 \end{vmatrix} = -2$, $A_{22} = |M_{22}| = \begin{vmatrix} 3 & -2 \\ -3 & 2 \end{vmatrix} = 0$, $A_{23} = -|M_{23}| = -\begin{vmatrix} 3 & -3 \\ -3 & 4 \end{vmatrix} = -3$

$A_{31} = |M_{31}| = \begin{vmatrix} -3 & -2 \\ -1 & -1 \end{vmatrix} = 1$, $A_{32} = -|M_{32}| = -\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix} = 1$, $A_{33} = |M_{33}| = \begin{vmatrix} 3 & -3 \\ 1 & -1 \end{vmatrix} = 0$

The cofactor matrix is

$$B = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 0 & -3 \\ 1 & 1 & 0 \end{pmatrix}$$

Hence,

$$\text{Adj } A = B^t = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & 1 \\ 2 & -3 & 0 \end{pmatrix}$$

Definition 2.6.4 If A is invertible, then $\det(A) \neq 0$ and

$$A^{-1} = \frac{1}{\det(A)}\text{Adj } A$$

Example 2.26 Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & -3 & -2 \\ 1 & -1 & -1 \\ -3 & 4 & 2 \end{pmatrix}$$

if exist.

Solution: $\det(A) = 3(-2+4) - (-3)(2-3) + (-2)(4-3) = 1$ and Adj $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & 1 \\ 2 & -3 & 0 \end{pmatrix}$. Then

$$A^{-1} = \frac{1}{\det(A)}\text{Adj } A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & 1 \\ 2 & -3 & 0 \end{pmatrix}$$

Example 2.27 Find the inverse of $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$

Solution: $A^{-1} = \frac{1}{2\times 3 - 5\times 1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$$B^{-1} = \frac{1}{-1\times 4 - 3\times(-2)}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & \frac{-3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix}$$

2.7 Systems of Linear Equations

An equation of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2.1}$$

are called systems of m linear equations in n unknowns. The above system of linear equation can be written as $AX = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Here A is called the coefficient matrix, X the unknown value, and b is the right hand side value (the constant). If $b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, the system is called homogeneous; otherwise it is called non-homogeneous. A homogeneous system has at least one solution, the trivial solution i.e. $x_1 = 0,\ x_2 = 0,\ ,\dots, x_n = 0$. A **solution** to a linear system $x_1,\ x_2,\ ,\dots, x_n$ is a sequence of n numbers $a_1,\ a_2,\ ,\dots, a_1$ such that each equation is satisfied when we substitute

$$x_1 = a_1,\ x_2 = a_2,\ ,\dots, x_n = a_1$$

The matrix $A|b = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$ is called the **augmented matrix**, for the system.

Definition 2.7.1 If a system of equations has at least one solution, it is said to be ***consistent***; if it has no solution, it is said to be ***inconsistent***. If a consistent system of equations has exactly one solution, the equations of the system are said to be **independent**; if it has an infinite number of solutions, the equations are called **dependent**.

2.7.1 Solving System of Equation using Cramer's Rule

Consider a system of linear equation consisting of n equations with n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad & \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{2.2}$$

(a) The system (2.2) has a unique solution if the determinant of the coefficient matrix $(D = \det(A))$ is nonzero. Hence, the solution is given by
$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$$
where D_k is the determinant obtained from D by replacing the k^{th} column in D by the column with the entries $b_1, b_2, \ldots, b_n$.

(b) If the system is homogeneous and $D \neq 0$, it has only the trivial solutions $x_1 = x_2 = \cdots = x_n = 0$. If $D = 0$, the homogeneous system has non-trivial solution.

Example 2.28 Solve the system

$$\begin{aligned} x - y &= 1 \\ -2x + 5y + z &= 3 \\ -x + 3y + z &= 2 \end{aligned}$$

Solution: The coefficient matrix is $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 5 & 1 \\ -1 & 3 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$$D = \det(A) = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 5 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 1, \quad D_1 = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 3$$

$$D_2 = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 3 & 1 \\ -1 & 2 & 1 \end{vmatrix} = 2, \quad D_3 = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 5 & 3 \\ 2 & 3 & 2 \end{vmatrix} = -1$$

Hence, $x = \frac{D_1}{D} = \frac{3}{1} = 3,\ y = \frac{D_2}{D} = \frac{2}{1} = 2,\ z = \frac{D_3}{D} = -1$

2.7.2 Solving System of Equation using Gaussian Elimination Method

The method is based on the idea of reducing the given system of equations $Ax = b$, to an upper triangular system of equations $Ux = z$, using elementary row operations.
Consider the system of equation

$$AX = b$$

To solve this system:

Step 1: Augment the coefficient matrix with the constant (RHS) value i.e., $A|b$

Step 2: Perform elementary row operation to change the augmented matrix into upper triangular matrix.

i. If $\text{Rank}(A) = \text{Rank}(A|b) = n$, then the system has a unique solution.
ii. If $\text{Rank}(A) = \text{Rank}(A|b) < n$, then the system has infinitely many solution.
iii. If $\text{Rank}(A) < \text{Rank}(A|b)$, then the system has no solution.

Step 3: Use back substitution to find the unknown values.

Example 2.29 Solve the system of linear equations

$$3x-2y+z=-1$$
$$5x-4y-z=3$$
$$-2x+y-z=2$$

■

Solution: Since , the solution to the given system is $x=-2, y=-3,$ and $z=-1$.

Example 2.30 Solve the system of linear equation by using Gaussian elimination methods.

$$\begin{aligned} x-3y+z &= 2 \\ 2x-y-z &= 9 \\ -3x+14y-6z &= -1 \end{aligned}$$

■

Solution: The coefficient matrix is $A=\begin{pmatrix} 1 & -3 & 1 \\ 2 & -1 & -1 \\ -3 & 14 & -6 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $b=\begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}$ Write the augmented matrix and apply row operations

$$A|b \sim \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 2 & -1 & -1 & 9 \\ -3 & 14 & -6 & -1 \end{array}\right) \sim \xrightarrow[R_3\longrightarrow R_3+3R_1]{R_2\longrightarrow R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 5 & -3 & 5 \\ 0 & 5 & -3 & 5 \end{array}\right)$$
$$\sim \xrightarrow{R_3\longrightarrow R_3-R_2} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 5 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The given system reduced to

$$\begin{aligned} x-3y+z &= 2 \\ 5y-3z &= 5, \quad 0=0 \end{aligned}$$

The system has infinitely many solution.
Let $z=t$

$\Longrightarrow 5y=5+3z \Longrightarrow y=1+\frac{3}{5}t$ and $x=2+3y-z=2+3(1+\frac{3}{5}t)-t=5+\frac{9}{5}t-t=5+\frac{4}{5}t$
where t is any real number. Hence,

$$x=5+\frac{4}{5}t,\ y=1+\frac{3}{5}t, \text{ and } z=t \quad \forall t\in\mathbb{R}$$

For example, if $t=0$ then $x=5,\ y=1,\ z=0,$ if $t=-5$ then $x=1,\ y=-2,\ z=-5$

Example 2.31 Solve the system of linear equation by using Gaussian elimination methods.

$$\begin{aligned} x-3y+z &= 2 \\ 2x-y-z &= 5 \\ -3x+14y-6z &= 1 \end{aligned}$$

■

Solution: The coefficient matrix is $A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -1 & -1 \\ -3 & 14 & -6 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ Write the augmented matrix and apply row operations

$$A|b \sim \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 2 & -1 & -1 & 5 \\ -3 & 14 & -6 & 1 \end{array}\right) \sim \xrightarrow[R_3 \longrightarrow R_3+3R_1]{R_2 \longrightarrow R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 5 & -3 & 1 \\ 0 & 5 & -3 & 7 \end{array}\right)$$
$$\sim \xrightarrow{R_3 \longrightarrow R_3-R_2} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 5 & -3 & 5 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

The given system reduced to

$$\begin{aligned} x-3y+z &= 2 \\ 5y-3z &= 5 \\ 0 &= 6(\text{ impossible}) \end{aligned}$$

Therefore, the given system has no solutions.

Solving a System of linear Equations Using Inverse Method

Consider the system of linear equation

$$Ax = b \tag{2.3}$$

where A is invertible square matrix. Multiply equation (2.3) by A^{-1} we get

$$x = A^{-1}b$$

which is the unique solution of the given system.

Example 2.32 Solve

$$\begin{aligned} 2x+4y-2z &= 6 \\ -4x-6y+z &= 1 \\ 3x+5y-z &= -1 \end{aligned}$$

Solution: The coefficient matrix is $A = \begin{pmatrix} 2 & 4 & -2 \\ -4 & -6 & 1 \\ 3 & 5 & -1 \end{pmatrix}$.

From example (2.23) the inverse of A is

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -3 & -4 \\ -\frac{1}{2} & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}$$

Using this result, we find that the solution of the given system is

$$X = A^{-1}b = \begin{pmatrix} \frac{1}{2} & -3 & -4 \\ -\frac{1}{2} & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3-3+4 \\ -3+2-3 \\ -6+1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -7 \end{pmatrix}$$

OR $x = 4, \quad y = -4, \quad z = -7$